Invited Lecture

What can History do for the Teaching of Mathematical Modeling in Scientific Contexts?

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ABSTRACT We will explore the role of history as a resource through which students can gain experience with authentic mathematical modeling in scientific contexts i.e. when mathematical modeling is used as a research tool, a practice, to gain knowledge in other areas. Three modeling episodes from the 20th century will be presented and analyzed with respect to modeling strategies, practices, items used in the modeling construction and cross-disciplinary epistemic issues — and an analytical framework for analyzing modeling episodes in scientific context will be presented. The framework will be discussed with respect to the modeling cycle in mathematics education, highlighting issues in the framework which are not featured explicitly in the modeling cycle. It will be illustrated how and in what sense history makes it possible to invite students into the work place of scientists that used and experimented with mathematical modeling as a research practice, i.e. its significance in creating such teaching and learning environments. Finally, the value of developing students' historical awareness for preparing them for tertiary studies where mathematical modeling might play a role will be discussed.

Keywords: History of mathematics; Mathematical modeling; Mathematics education.

1. Introduction

History can serve a variety of purposes in mathematics education[2] — one of them, which is the focus of the present talk, is to provide a window for students into "mathematics in the making" so to speak (Kjeldsen, 2018). In the following, this role of history will be explored with respect to the teaching and learning of mathematical modeling in scientific contexts — and with this I mean, when mathematical modeling is used as a research tool, as a research practice, to gain knowledge in other areas.

On the one hand, mathematical modeling has come to play an important role in scientific practices during the 20th century, and modeling has also by now been included in mathematics curricula in many countries (Blum et al., 2007). Being aware of the role of mathematical modeling in scientific context, both in mathematics

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[2] For a recent overview, see (Clark et al. 2020)

education and in other science educations, might open students' eyes for the possibilities, mathematical modeling has to offer in other disciplines as a research tool.

On the other hand, this is neither easy nor unproblematic. Scientists from different disciplines have different views on what counts as a "good" model, as a "good" explanation, as useful knowledge and what is relevant knowledge. However, to become acquainted with such differences in high school and undergraduate science education, might encourage and prepare students for interdisciplinary studies and collaboration.

In the following, I will argue that and illustrate how history can contribute to making students aware of issues of how mathematical modeling can function as a research practice in science. First I will present and analyze three modeling examples from the history of mathematics from the 20th century: John von Neumann's model of general equilibrium in economics, Vito Volterra's predatory-prey model, and Nicolas Rashevsky's model of cell division. In each of these cases there were reactions and discussions from scientists from the target domain. The work of these three authors together with the reactions and discussions with scientists from the target domain, provide a possibility to bring the actors' "voices" into the classroom. Or, if we look at it from the teaching side, a possibility to invite students into the work place of scientists that used and experimented with mathematical modeling as a research practice. Secondly, I present a framework for analyzing and comparing modeling episodes in scientific contexts to understand modeling strategies, practices and cross-disciplinary epistemic issues.[3] I will discuss the analyses with respect to the modeling cycle in mathematics education, pointing out shortcomings in the sense of elements in the framework, which are not featured explicitly in the modeling cycle. A student project work will be presented to illustrate how students, working with the Rashevsky case, were invited into an authentic modeling workshop. Finally, the value of developing students' historical awareness for preparing them for tertiary studies where mathematical modeling might play a role will be discussed.

2. Case 1: John Von Neumann's Model of General Economic Equilibrium

John von Neumann was born in Budapest in 1903. He immigrated to the USA where he became a professor at the Institute for Advanced Study in Princeton. In 1932 he gave a talk in Princeton where he presented a mathematical model in economics. It was published five years later in Karl Menger's *Ergebnisse eines mathematischen Kolloquiums.* In 1945 it was translated into English with the title *A Model of General Economic Equilibrium*. Von Neumann considered a general economy where there are n goods $G_1, \dots, G_n$ which can be produced by m processes $P_1, \dots, P_m$. He asked the question "Which processes will be used (as "profitable") and what prices of the goods will obtain?" (von Neumann, 1937, p. 75).

[3] This framework was first presented in Jessen and Kjeldsen, 2021.

He mathematized this economy by setting up a system of six linear inequalities that express relationships between the intensities of the processes, which are represented by $x_1, \dots, x_m$ and the prices of the goods, which are represented by $y_1, \dots, y_n$ (von Neumann, 1937, p. 75−76). The two parameters α and β represent the expansion factor and the interest factor, respectively. The coefficients a_{ij} and b_{ij} represent the amount of the good G_j used in the process P_i, and the quantity of the good G_j produced by the process P_i, respectively.

$$x_i \geq 0$$

$$y_j \geq 0$$

$$\sum_{i=1}^{m} x_i > 0$$

$$\sum_{j=1}^{n} y_j > 0$$

$$\alpha \sum_{i=1}^{m} a_{ij} x_i \leq \sum_{i=1}^{m} b_{ij} x_i \text{ for all } j$$

$$\beta \sum_{j=1}^{n} a_{ij} y_j \geq \sum_{j=1}^{n} b_{ij} y_j \text{ for all } i$$

The first four inequalities are self-explanatory, the fifth one makes sure that we do not consume more of the good G_j than is produced in the economy. And the last one means that there is no profit in the system — everything gets re-invested.

In order to solve this model von Neumann wanted to investigate whether a solution exists, and he was able to prove the existence of a solution to the inequality system. In order to do so, he first transformed the problem into a problem of a saddle point for a certain function. He then proved a new mathematical result, a generalization of Brouwer's fixed point theorem, which he then used to prove the existence of a saddle point and thereby the existence of a solution to the inequality system (see Kjeldsen 2001). With this result, von Neumann had proved that such a general economy has an equilibrium.

It was not a constructive proof it was an existence proof, so von Neumann did not construct a solution. In 1945 when the paper was translated into English by the British economist David Champernowne, Champernowne wrote a note where he raised several critical aspect to the use of von Neumann's model in economics:

"Approaching these questions as a mathematician, Dr. Neumann places emphasis on rather different aspects of the problem than would an economist. […] The paper is logically complete […]. But at the same time this process of abstraction inevitable made many of his conclusions inapplicable to the real world […] the reader may begin to wonder in what way the model has interesting relevance to conditions in the real world. […] utmost caution is needed in drawing from them any conclusions about the determination of prices, production or the rate of interest in the real world." (Champernowne, 1945, pp. 10-15)

As Chapernowne's warning here indicates there is not necessarily an agreement about what constitutes a "solution", when mathematics is used in scientific practices in other domains – it is context dependent. The disciplinary lens that is used, especially when new modes of inquiry are under development, plays a significant role in the acceptance or non-acceptance of modeling results. Despite Champernowne's critic, and other critical voices in economics, von Neumann's paper is considered a rather important paper, and it has played a significant role in the development of theory in economics, see e.g. (Dore et al., 1989). It is an example of models as elements of economic theories. The case also illustrates that how modeling and models are perceived and judged depend on the recipient's conception of the purpose: for von Neumann, the purpose was to prove consistency; does an equilibrium, a solution to the linear inequality system, exist or not? That was the interesting question. Champernowne though, wanted to solve concrete (economic) problems in practice and here von Neumann's model didn't really help.

3. Case 2: Vito Volterra's Predatory-Prey Model

The second case is Vito Volterra's now well-known predator-prey model. Volterra was born in Italy in 1860. He became a professor of mechanics at the University of Turin, and of mathematical physics at University of Rome in the year 1900. He was asked by the biologist, Umberto D'Ancona, if he could explain the observation that the reduced fishing in the Upper Adriatic during WWI, in contrast to what one might think, apparently was more favorable for the predator fish than for the prey (Volterra, 1926). Volterra approached the phenomenon as if it was a problem in mechanics by e.g. neglecting friction from the environment. He explained his approach in a paper published in 1927 where he wrote that:

"To facilitate the analysis it is convenient to present the phenomenon schematically, by isolating those factors one wishes to examine, assuming they act alone, and by neglecting the others. […] I have started by studying only the intrinsic phenomena due to the voracity and fertility of the coexisting species." (Volterra, 1927 [1978, p. 68])

Volterra only took the predatory and fertility into account, and constructed a hypothetical system based on these two kinds of events. He further assumed that the two populations of fish developed continuously, because he wanted to use the theory of differential equations in his modeling. He further assumed that the birth rate of the prey (ε_1) is constant, so they grow exponentially if they live alone, and he assumed that the number of predators will decrease exponentially in the absent of prey, with ε_2 denoting the death rate. To model the predation, he used a mechanical analogy, which he called the "method of encounters". He envisioned that encounters between two competing species, N_1 and N_2, occur at random as with particles in a perfect gas in a closed container, so the predation is proportional to the product of the numbers of species, that is their densities, so to speak. Based on this analysis, he derived the differential equations below and we can see that in the second set of equations, the encounters between the species are implemented. These are now known as the "Lotka-Volterra" equations (Volterra1927 [1978, p. 80, 95]):

$$\frac{dN_1}{dt} = \varepsilon_1 N_1, \quad \frac{dN_2}{dt} = -\varepsilon_2 N_2,$$

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2)N_1, \quad \frac{dN_2}{dt} = (-\varepsilon_2 + \gamma_2 N_1)N_2,$$

Volterra represented the solutions in a graph, showing the now well-known periodic behavior of the two populations, which his model was able to capture. His model was also able to account for the observation of D'Ancona, that the reduced fishery during World War I was more favorable for the predatory fish than for the prey fish.

Volterra was concerned about the validation of his model through empirical data. But here D'Ancona had a different point of view, which he expressed very clearly in a letter he wrote to Volterra in 1935, where he stated that:

"My observations [of the fisheries in the Upper Adriatic] could be interpreted in the sense of your theory, but this fact is not absolutely unquestionable: it is only an interpretation. …You should not think that my intention is to undervalue the experimental research supporting your theories, but I think that it is necessary to be very cautious in accepting as demonstrations these experimental researches. If we accept these results without caution we run the risk of seeing them disproved by facts. Your theory is completely untouched by this question. It lay on purely logical foundations and agrees with many well-known facts. Therefore it is a well-founded working hypothesis from which one could develop interesting researches and which stands up even if it is not supported by empirical proofs" (D'Ancona to Volterra 1935, quoted from Israel, 1993, p. 504).

There is a discussion of epistemic value in this letter. D'Ancona was, which he expressed very clearly, of the opinion that the exploration of a mathematical model that has been derived from a concrete phenomenon though based on crude, simplifying hypotheses and idealizations, can lead to new (valuable) insights even if the model cannot be confirmed by data. D'Ancona's view in this matter has been interpreted by the Italian historian of science Georgio Israel (Israel, 1993) as indicating a shift towards a more modern abstract conception of modeling.

4. Case 3: Nicolas Rashevsky's Early Model on Cell Division

The third and last case is Nicolas Rashevsky's early work on cell division. He was born in 1899 in Chernigov in Ukraine. He held a doctorate in theoretical physics from the University of Kiev. He immigrated to the USA, first to Pittsburg in 1924 where he came to work at the Research Department of the Westinghouse Electric and Manufacturing Company. In 1934 he moved to University of Chicago, to academia, through a fellowship from the Rockefeller foundation (Abraham, 2004).

Rashevsky's ambition was to build a mathematical biology on a physico-chemical basis — he wanted to mimic the development of physics based on mathematics. He saw the role of mathematics as a 'gateway', he said, to the "hidden fundamental properties of nature" (Rashevsky, 1935, p. 528). Rashevsky was very much outspoken and he wrote many papers where we can follow his modeling and also his thoughts and philosophy of mathematical modeling. Here is how he expressed it in Nature in 1935:

"… very little attempt has been made to gain an insight into the physico-chemical basis of life, similar to the fundamental insight of the physicist into the intimate details of atomic phenomena. Such an insight is possible only by mathematical analysis; for our experiments do not and cannot reveal those hidden fundamental properties of Nature. It is through mathematical analysis that we must *infer*, from the wealth of known, relatively coarse facts, to the much finer, not directly accessible fundamentals." (Rashevsky, 1935, p. 528).

Here he was questioning the experimental practice in biology. He wanted to have a more theoretical practice, and he promoted the use of what he called 'paper and pencil models', which he had explained in the journal *Physics* in 1931:

"… a physicist has enough confidence in the results of his calculations, that he does no need actually to build a model, and may satisfy himself by investigating mathematically, whether such a model is possible or not. The value of such "paper and pencil" models is not only as great as that of actual "experimental" models, but in certain respects it is even greater. The mathematical method has a greater range of possibilities, than the experimental one, the latter being often limited by purely technical difficulties." (Rashevsky, 1931, p.143–153)

Rashevsky presented his model of cell division to the biologists at a symposium that was held on Long Island, New York in 1934. Here he confronted the biologists, asking the question: "Do we need to assume some special independent mechanisms to explain cell division?" And he gave the answer: No. "Cell division can be explained as a direct consequence of the forces arising from cell-metabolism" (Rashevsky, 1934, p. 188). He also gave the 'recipe' for how that can be done, namely logically and mathematical from a set of well-defined general principles. … and he claimed the superiority of mathematics:

"… it is only natural to assume that the lack of our knowledge of the fundamental causes of biological phenomena, in spite of the tremendous amount of valuable facts, is due to the lack of use of deductive mathematical methods in biology" (Rashevsky, 1934, 188−198).

In his modeling, he drew an analogy to work, he had done while he was at Westinghous. There he had worked on dynamics of colloid particles and division of droplets. He linked this to cell division by conceptualizing a cell as a physical system which is liquid, and from this he derived that, due to metabolism:

"… there will be a difference in concentration outside and inside the system, the concentration outside being greater. […] We have to do with a phenomenon of diffusion governed for a quasi-stationary state by the equation

$$D\Delta^2 c = q(x, y, z)$$

where D denotes the coefficient of diffusion, c the concentration, and $q(x, y, z)$ the rate of consumption of the substance …" (Rashevsky, 1934, p. 189).

This was the first step in his modeling. The next step was to investigate at the level of molecules, so he derived expressions for the forces produced by a gradient of concentration. He calculated the force exerted on a molecule (A) of the solvent by all molecules (B) of the solute. By integration, he derived an expression f_s for the force exerted on each element of volume of the solvent by the solute, and he also calculated the force f_o acting on each volume as a result of osmotic pressure, and the force f_r of repulsion between molecules. He summarized, saying that:

"…we see that a gradient of concentrations produces a force per unit volume which is the sum of the above three forces [$f_s + f_o + f_r$]" (Rashevsky, 1934, p. 191).

The third step in his modeling was to make further idealizations. He assumed that the cells were homogenous and spherical. He emphasized that he was aware that this is not how cells look like, but that this idealization would give a general qualitative picture. He calculated that under these assumptions, when a cell divides, the volume energy will decrease and the surface energy will increase, and for large radius of the cell, the increase will be less than the decrease. Then he invoked the principle of free energy from physics, and argued that:

"As any system tends to assume such a configuration, for which its free energy has the smallest value possible, one is tempted to infer that … division of a cell will occur spontaneously as soon as, … the cell will exceed the critical size" (Rashevsky, 1934, p. 192).

Well, he was well aware, as he said, that "unfortunately […] things are not so simple", so he only concluded that:

"every cell, by virtue of the processes of metabolism … contains in itself the necessary conditions for spontaneous division above a certain size" (Rashevsky, 1934, p. 192).

The talks from the symposium are published in a proceedings together with the discussions after the talks, so we can follow the discussion between the biologists and Rashevsky after Rashevsky's talk. It is quite clear from the discussion that the biologists didn't approve of Rashevsky's method. They wanted to know what "example in nature would be nearest to this theoretical case?" (Rashevsky, 1934, p. 195). If we look at Rashevsky's assumptions, we can see that they range from speculation to adjustments for the sake of mathematics and the biologists questioned all these assumptions.

From Rashevsky's point of view, his modeling fulfilled his purpose of investigating *possible* explanations for cell division by deducing consequences and compare them with empirical results. But it did not fulfill the biologists' purpose. They wanted to know *the* mechanism of cell division not all kinds of imaginative possibilities. According to the biologists, Rashevsky's approach lacked reality and experiments. So here, we see very clearly this clash of practices across disciplinary boundaries with biology on the one side and the mathematical physical domain on the other side (Kjeldsen, 2019).[4]

5. Analysis and Comparison of The Cases: Developing A Framework

Tab. 1 below (on the next page) represents a (preliminary) analytical framework for analyzing and comparing modeling episodes in scientific contexts to understand modeling strategies, practices and cross-disciplinary issues. The framework is constructed with inspiration from Boumans' (2015) "model" for modeling, Gelfert's (2018) ideas of explorative modeling supplemented by the notions of modeling strategy and epistemic value, issues which were illuminated by the analyses of the historical actors' actions and disputes in the three episodes. To be more specific: the cases are analyzed and compared with respect to "meta aspects", "items used in the modeling construction" and "explorative model function". The meta-aspects are divided into 'motivation', 'strategy' and 'discussion of epistemic value'. The items used in the modeling construction that have been identified in the analyses are 'analogies', 'mathematical concepts and theories', 'theoretical notions from other areas' and 'empirical data'. Finally, the explorative model functions are the three functions 'starting point', 'proof of principle', and 'possible explanation' as identified by Gelfert. They will be explained below.

Regarding meta-aspects, the analyses of the three cases with respect to the motivation, strategy and the discussions of epistemic values for the modeling, we find that there are various differences. In von Neumann's case, the motivation was to develop theory, in Volterra's case, it was to explain an observed phenomenon, a pattern, and in Rashevsky's case, the purpose of the modeling was to search for causality, to explain cell division in terms of physics and chemistry. If we look at the strategies, we see that von Neumann created and analyzed an abstract mathematical structure, which was his model. Volterra, he developed a hypothetical system from the kinematics of gases, and Rashevsky conceptualized a cell as a liquid system that transforms substances. Comparing the discussions of epistemic value of the model results, we found that in von Neumann's case, there was a discussion between the epistemic value of internal consistency versus the lack of reality; in Volterra's case there was a discussion between Volterra and D'Ancona about whether it was necessary to have the model verified by data or whether it had epistemic value in itself that the model was founded on logical foundation. In Rashevsky's case, there was the discussion of the epistemic value of a possible explanation, as Rashevsky cherished — while the biologists considered this as just pure imagination, which they deemed irrelevant.

If we look at the items that went into the modeling cases there are at least four items that can be identified: analogies played a role that had an effect on the modeling.

[4] For a scientific biography of Rashevsky, see Shmailov, 2016. For issues of interdisciplinary collaboration, see also Keller, 2002.

In Volterra's case it was the collision of molecules, and Rashevsky made the analogy to droplets of liquid in physics. These analogies effected the way the models were constructed. It was through these analogies that Volterra and Rashevsky set up their models.

Tab. 1. Analytical framework for analyzing mathematical modeling episodes in scientific contexts (Jessen and Kjeldsen, 2021).

		Von Neumann	Volterra	Rashevsky
Meta aspects	**Motivation**	Develop economic theory.	Explain a concrete phenomenon due to reduced fishing.	Explain cell division in terms of physics and chemistry.
	Strategy	Abstract mathematical structure of a general economy.	Simple hypotheses, simplifications and idealizations.	Conceptualized a cell as a liquid system that transforms substances.
	Discussion epistemic value	Existence of solution/ internal consistency vs, lack of reality/not useful.	Verification through data vs. purely logical foundation.	Possible explanation, promising vs. imaginary causes, irrelevant.
Items used in the modeling construction	**Analogies**		Collisions of molecules.	Physical phenomenon of droplets.
	Mathematical concepts/theories/ techniques	Linear inequalities, fix-point techniques.	Calculus, systems of differential equations.	Differential equation (diffusion equation), integration.
	Theoretical notions from other areas		Method of encounters.	Physical forces, surface/volume tension, free energy, principle of free energy.
	Empirical data		Served as motivation not as verification.	Served as control not as verification.
Explorative model function	**Explorative function 1., 2., 3.**	2. Proof of principle.	2. Proof of principle. 3. Possible explanation.	1. Starting point. 3. Possible explanation.

Also the mathematics they chose had an influence: for von Neumann it was systems of linear inequalities, and he used fixpoint techniques to solve the model, Volterra used differential equations, and Rashevsky used the diffusion equation and integration techniques. Notions from other areas were implemented into the target domain and influenced the model constructions. In Volterra's case it was the 'method of encounter', and Rashevsky used the notion of force, surface tension and the principle of free energy from physics. Data played different roles in the three episodes: in von Neumann's case data wasn't really present, in Volterra's case it served as a motivation. He wanted it also to be necessary for verification, but here D'Ancona's view was that he didn't think that was necessary. In Rashevsky's case the data didn't play a role for verification it only played a role as control. He compared his theoretical results of the model with experimental results to check the order of magnitude of the radius for when a cell divides.

The notion of explorative modeling was introduced by Axel Gelfert (Gelfert, 2018). By explorative modeling he means:

"Models …that allow us to extrapolate beyond the actual, thereby allowing us to also explore possible […] scenarios" (Gelfert, 2018, p. 253).

He continues, explaining that

"The use of models, then, is not restricted to the initial goal of representing actual target systems. ... [some] models only aim to provide potential explanations of general patterns, […] without thereby claiming to be able to identify the actually operative causal or explanatory factors" (Gelfert, 2018, p. 253).

Gelfert sees explorative function as one of the key functions of mathematical modeling as a research practice to gain new insights.

He distinguishes between three functions of explorative models. One function is that it can aim at a "starting point". This is exploration in the hope of finding fruitful ways to proceed — in the absence of a well-formed underlying theory (Gelfert, 2018, p. 254). Rashevsky's case very much fits this description. He didn't have a starting point, and he explored the model in terms of that. The second function, Gelfert identified is what he calls "proofs of principles". Here Gelfert uses Volterra as a case, and he says that in Volterra's case we see that "the methodology of differential equations is suitable for generating insights into the dynamics of (discrete) populations" (Gelfert, 2018, p. 257). So, Voterra's model and exploration constituted a proof of principles. And I think that also in von Neumann's case, the modeling can be identified as having this explorative function. It constitutes a proof of the existence of equilibrium of such a general economy. The third explorative modeling function is to come up with "potential explanations". We saw that function very clearly in Rashevsky's case, but also in Volterra's case.

6. Significance of History in The Teaching of Mathematical Modeling

In mathematics education there are various "models" of modeling and the modeling cycle features prominently in this literature. The figure below is one example of a modeling cycle (Fig. 1). It is adapted from Blomhøj and Jensen (2007). It is an analytic model of a modeling process. Our analyses of the modeling constructions in the three cases brought out elements such as the motivation, the underlying agenda, of the modeler, the modeling strategy, the use of analogies, the effect of the chosen mathematics on the construction of the model, the possible import of theoretical notions from other disciplines and their effects on the model construction, the explorative function of the model, the issues of epistemic value of model results across disciplinary boundaries. These elements are not featured explicitly in the modeling cycle.

Another account of modeling is Boumans' work in history and philosophy of economics from 2005, which has been used in the construction of the framework in Tab. 1. Based on historical case studies from history of economics, he developed a conception of modeling as "baking a cake without a recipe" (Boumans, 2005, p. 16).

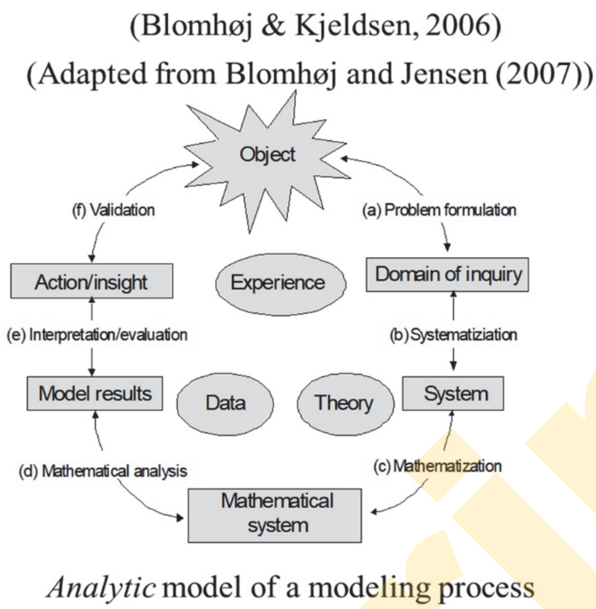

Fig. 1. An analytic model of a modeling process adapted from Blomhøj and Jensen, 2007

He conceives models as being built by fitting together various elements from different sources. The elements, he brings up, are theoretical notions, mathematical concepts, mathematical techniques, stylized facts, empirical data, policy views, analogies, metaphors. In his "model" of modeling, Boumans try to capture what goes into modeling in practice, like mixing the pieces together. So there is a different focus in Boumans' conception of modeling than in the modeling cycle, and it has these various ingredients and tools, which are made explicit. Boumans' model might be a valuable supplement to the modeling cycle in mathematics education.

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However, the epistemic disciplinary issues — the different practices, the epistemic values and explorative functions that were displayed in the analyses of the historical cases are not captured explicitly in this model either. This is where the historical cases can serve a function in the teaching and learning of mathematical modeling in scientific practices. Historical episodes represent cases where the actors get a voice and they can function in that way in the teaching of mathematical modeling in scientific practice to elicit such issues and to make them explicit objects of students' reflections. They can serve as "invitations" of students into "modeling in the making". Below, I will present an example where this happened in teaching, where a group of students worked with Rashevsky's modeling of cell division that he presented at the Cold Spring Harbor Symposium.

The case of Rashevsky was used in a project work with third semester undergraduate students in an interdisciplinary Bachelor of Science program at Roskilde University in Denmark in 2003. At Roskilde University, problem-oriented teaching where students work together on a project in groups is a pedagogical corner stone in all study programs. In the interdisciplinary Bachelor of Science program, the students work half of the study time throughout a semester in groups of 3–8 students with a bigger project supervised by a professor.

In the spring semester of 2003, there was a group of four students that worked with the Rashevsky case. The problem that guided the students' work in the project was the question, why Rashevsky was unable to get through to the biologists of his time with his ideas. (Andersen et al., 2003, p.2).

So what did they do? The students read and discussed these early works by Rashevsky on cell division including the paper of the talk he gave at the Cold Spring Harbor Symposium. They also read articles and literature from the history and philosophy of science.

In the learning environment that was created through the problem-oriented project work, the students were invited into Rashevsky's "workshop". They obtained access to a modeling process at the frontier of (former) research. The students constructed illustrations of Rashevsky's modeling to support their understanding of how he found expressions for the various forces, that he calculated. In this process, through their work with Rashevsky's paper, the students gained hands on experiences with the mathematization process and techniques of adding up by integration, see (Kjeldsen and Blomhøj, 2009).

The disagreement between Rashevsky and the biologists, that the students studied in connection to Rashevsky's talk, supported their competencies regarding interpretation and validity of the results produced by a model. In this sense, their work with the historical sources enhanced and developed the students' modeling competency. What the students also gained was that they obtained insights into aspects of the emergence (and struggles) of interdisciplinary fields of research. The historical case gave the students concrete examples of assumptions and beliefs underneath research processes about how the world function, and illustrated explicitly for them that such assumptions guide the questions one asks and the kind of answers one can acquire (Kjeldsen, 2017).

7. Conclusion: What can History Do for The Teaching of Modeling?

The analyses of the three historical episodes of mathematical modeling in scientific contexts and the reception of the models as research practice in the target domain identified essential factors in modeling constructions in scientific contexts, which are displayed in the analytical framework in Tab. 1. As we have shown in (Jessen and Kjeldsen, 2021), the relation between mathematical modeling in scientific context and upper secondary education is, at least in Denmark, very vague. Our analyses of curriculum in high school showed that "the knowledge to be taught reflects to a minor degree the nature of modeling and practices […] found in the historic cases" (Jessen and Kjeldsen, 2021, p. 54).

Through the historical cases, students can be invited into the workshop of scientists and follow mathematical modeling "in the making", and thereby gain experience with various modeling strategies, the explorative nature of modeling and the role of e.g. analogies, the chosen mathematical theory and techniques, the implementation of theoretical notions from other areas on the modeling process, the model and its reception — all issues that are essential for modeling as a research practice.

Investigation of such debates from history of science in mathematics education brings the voices of authentic actors into the classroom, and can raise students' awareness and understanding of methodological issues and debates in interdisciplinary scientific research of today. It illustrates the uncertainty inherent in research at the frontier, where new areas are explored and/or new methods are employed. It also promotes students to reflect about the uses of mathematics to obtain knowledge in other areas, and make students see how, what seems to be a valid scientific approach in one field, can be rejected by experts from a different field — or researchers with a different perspective, see also (Green and Andersen, 2019).

More generally, history can bring authenticity into the teaching and learning of mathematics — it is a source of authentic mathematical (modeling) activities. Chinn and Hmelo-Silver interpret what they call 'authentic inquiry' as "activities that scientists engage in while conducting their research" (Chinn and Hmelo-Silver, 2002, p. 171). Such activities are not so easy to implement neither in the teaching of mathematics as a scientific subject in itself nor in mathematical modeling in scientific contexts (or as professional task), as Frejd and Bergsten (2016) have discussed in a recent paper. Here history of mathematics serves a role qua being history. Episodes from history of mathematics, like the three that has been presented here, can provide a window into mathematics and mathematical modeling "in the making", so to speak. By using historical sources and episodes to invite students into the workshop of past scientists, a learning environment can be created where students can gain insights into and be challenged to reflect explicitly about how scientists get ideas for using modeling to explore research agendas in other areas, which strategies they use, the significance of various choices they make, how they argue, and how they learn. Students may also, as we have seen, come to reflect upon discussions and opinions about what counts as valid arguments, as useful knowledge among the various groups of actors, and realize that there might be differences here. In this sense, they also come to reflect upon the epistemology and the nature of mathematical modeling in scientific contexts. A learning environment that structures and promotes such kinds of reflections is an example of what we have called an 'Inquiry-reflective learning environment' (Kjeldsen, 2018), (Johansen and Kjeldsen, 2018), and by integrating historical cases in the teaching and learning of mathematical modeling it is possible to set up such inquiry-reflective learning environments.

I will finish by drawing attention to the notion and significance of developing students' historical awareness more broadly in mathematics education. The notion of historical awareness is based on the circumstance that both the past and the future are present in the present. The past as recollection and interpretations (of the past) and the future as a set of expectations. To develop students' historical awareness means to motivate their interest in and ability to ask questions about the past in order for them to gain an understanding of the complex world they live in.

As we have seen, working with historical episodes in an inquiry-reflective learning environment can provide access into people's/mathematicians'/scientists' creation of mathematical knowledge and modeling and/or their thoughts about it and their work-life opportunities — illustrating that it is a process that is constrained by the past and that it sets the possibilities for the creation of future mathematical knowledge and uses of mathematics (Kjeldsen, Clark and Jankvist, forthcoming). To come back to the teaching example presented above. Through their work with the Rashevsky-project, the students developed historical awareness with respect to the role of mathematical modeling in the sciences, and they obtained insights into aspects of the emergence (and struggles) of interdisciplinary fields of research. This helped them to orient themselves with respect to mathematics in their further education. So more generally, the purpose of bringing historical awareness into the mathematics classroom is also to enlighten students, and to give them tools to reflect on their own abilities and possibilities in and with mathematics in their future lives.

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